Fourier Transform

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  - Fourier Transform.
  - Discrete Fourier Transform.
- Satu Alaoutinen
  - Properties of 2-D Fourier Transform.
- Albert Sadovnikov
  - The Fast Fourier Transform.

Presentation Plan

- Introduction to Fourier Transform
- Examples of FT
- The Discrete Fourier Transform (DFT)
- Two Dimensional Transform

The FT is used in:

- linear systems analysis,
- antenna studies,
- optics,
- random process modelling,
- probability theory,
- quantum physics,
- boundary-value problems
- it has been very successfully applied to restoration of astronomical data
- wide range of application in image processing problems

Introduction to FT(1)

The *Fourier transform*, in essence, decomposes or separates a waveform or function into sinusoids of different frequency which sum to the original waveform. It identifies or distinguishes the different frequency sinusoids and their respective amplitudes.

Example

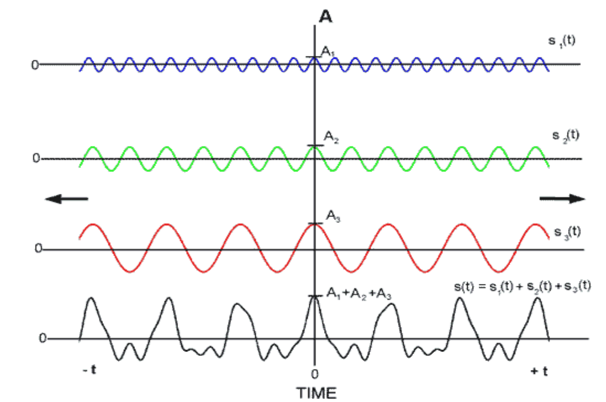

Figure 1. Three cosine waves with amplitudes A1, A2, and A3 combine to form a composite signal with amplitude A1 + A2 + A3.

Introduction to FT(2)

- ✓ The Fourier Transform is an important image processing tool which is used to decompose an image into its sine and cosine components.
- ✓ The output of the transformation represents the image in the *Fourier* or frequency domain, while the input image is the spatial domain equivalent. In the Fourier domain image, each point represents a particular frequency contained in the spatial domain image.
- ✓ The Fourier Transform is used in a wide range of applications, such as image analysis, image filtering, image reconstruction and image compression.

Frequency and Special Domains

- ✓ For simplicity, assume that the image I being considered is formed by projection from scene S (which might be a two- or three-dimensional scene, *etc.*).

Thus,

- ✓ The *frequency domain* is a space in which each image value at image position F represents the amount that the intensity values in image I vary over a specific distance related to F.
- ✓ In the frequency domain, changes in image position correspond to changes in the *spatial frequency*, (or the rate at which image intensity values) are changing in the spatial domain image I.

Example of Frequency and Special Domains

Let us suppose that there is the value 20 at the point that represents the frequency 0.1 (or 1 period every 10 pixels). This means that in the corresponding spatial domain image I the intensity values vary from dark to light and back to dark over a distance of 10 pixels, and that the contrast between the lightest and darkest is 40 gray levels (2 times 20).

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How it works?

Let *f(x)* be a continuos function of a real variable *x*.

- The FT of *f(x)* is defined as:

$$\Psi\{f(x)\} = F(u) = \int_{-\infty}^{+\infty} f(x) exp[-j2\pi ux] dx,$$
$$where \quad j = \sqrt{-1}$$

- Given *F(u)*, *f(x)* can be obtained by using the *Inverse FT:*

$$\Psi^{-1}\{F(u)\} = f(x) = \int_{-\infty}^{+\infty} F(u) exp[j2\pi ux] du.$$

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How it works?

✓ The FT of real function is generally complex and can be presented as

$$F(u) = R(u) + jI(u)$$

where *R(u)* and *I(u)* are real and imagery components of *F(u)*. Thus we can re-write equation in the following form:

$$F(u) = |F(u)|e^{j\phi(u)},$$
$$where \quad |F(u)| = [R^2(u) + I^2(u)]^{1/2}$$
$$and$$
$$\phi(u) = tan^{-1}\left[\frac{I(u)}{R(u)}\right].$$

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How it works?

✓ The magnitude function |*F(u)*| is called the Fourier spectrum of *f(x)* and *ϕ(u)* its *phase angle (phase).*

✓ The square of the spectrum is commonly referred to as the *power spectrum*. The term *spectral density* also is commonly used to denote the power spectrum.

$$P(u) = |F(u)|^2 = R^2(u) + I^2(u).$$

✓ The variable ***u*** appearing in the FT often is called the *frequency variable*

✓ This name arises from expression of the exponential term, - $exp[-j2\pi ux]$ using Euler's formula – in the form:

$$exp[-j2\pi ux] = Cos2\pi ux - jSin2\pi ux$$

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Conditions.

- The FT pair exists if f(x) is continuous and integrable and F(u) is integrable.

Which means that it exists for all real life signals.

Example

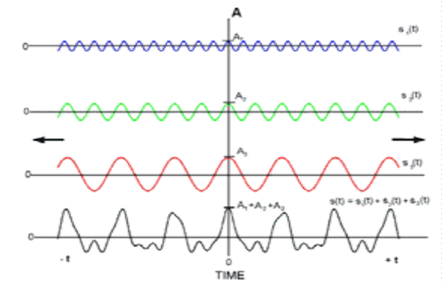

Figure 1. Three cosine waves with amplitudes A1, A2, and A3 combine to form a composite signal with amplitude A1 + A2 + A3.

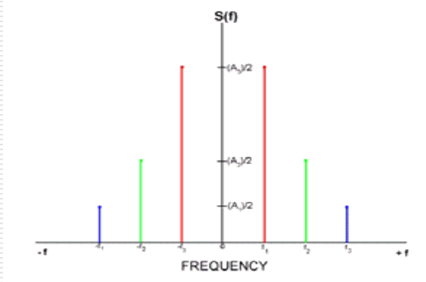

Figure 2. Fourier transform of 3-cosine composite signal in Figure 1 yields three pairs of real, even delta functions with corresponding amplitudes A1/2, A2/2, and A3/2.

Example

Since the original signal is real and even the Fourier transform must be real and even. Three pure cosine oscillations summate to make up s(t) so only three spectral lines are present in the Fourier transform, S(f). These spikes can be represented by Dirac delta functions that are functions of frequency, not of time.

Positive and Negative Frequencies

FT($A_1\cos 2pf_1t$) is $\Im\left[A_1 \cos 2\pi f_1 t\right] = \frac{A_1}{2}\delta(f+f_1) + \frac{A_1}{2}\delta(f-f_1).$

There are values (spectra lines) at both positive and negative frequencies

In this case they appear where the delta functions are non-zero,
i.e., where their arguments are zero,
at $f = +f_1$ and $f = -f_1$.

The concept of negative frequencies is critical for practical applications of digital processing in the frequency domain.

Concept of Negative Frequencies

- The idea of negative frequencies is to visualize a wheel rotating in one direction and then reversing the direction. Rotating in say the counterclockwise (CCW) direction illustrates positive frequency and clockwise (CW) rotation describes negative frequency. The rotating wheel view is a perfectly correct way of interpreting the + and - frequencies of the complex Fourier spectrum.

FT of A_1 Cos ($2pf_1t$)?

$$e^{i\phi} = \cos\phi + i\sin\phi,$$

from which we can write

$$A_1 e^{i2\pi f_1 t} = A_1 \cos(2\pi f_1 t) + i\, A_1 \sin(2\pi f_1 t).$$

The inverse of Euler's relation allows us to express the trigonometric functions as

$\cos(2\pi f_1 t) = \frac{A_1}{2}(e^{i2\pi f_1 t} + e^{-i2\pi f_1 t})$ and $\sin(2\pi f_1 t) = \frac{-iA_1}{2}(e^{i2\pi f_1 t} + e^{-i2\pi f_1 t}).$

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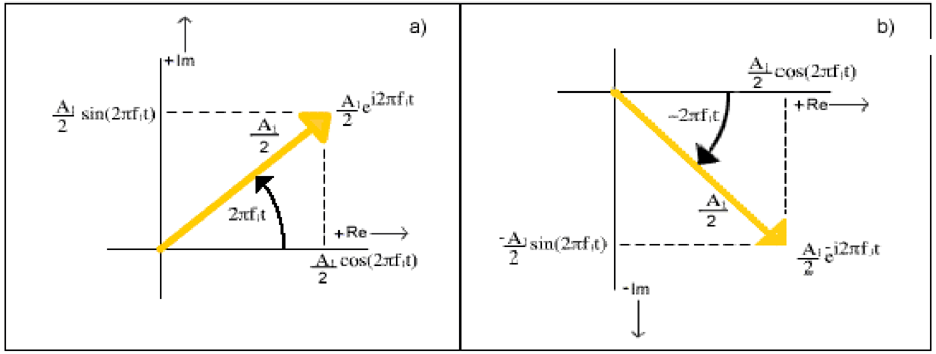
FT of A_1 Cos ($2pf_1t$)?

$A1/2 \exp\{i2pf_1t\}$ ={real and imaginary parts}=
$A1/2 \cos(2pf_1t) + A1/2 \sin(2pf_1t)$



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- The key observation is that since f = $\pm 2pf_1t$ is an angle that varies linearly with time, the vectors exp{$\pm$i2p f_1t} also vary with time.
  - ✓ at t = 0 the exp{$\pm$i2p f_1t}vectors lie along the positive, horizontal axis;
  - ✓ at a time t = 1 the exp{+i2p f_1t} vector has rotated through a CCW angle of $2pf_1$.
  - ✓ At an arbitrary time t, the rotating exp{+i2p f_1t} vector has an angle of $2pf_1t$ CCW from the positive real axis. The e{-i2p f_1t} vector, with the negative exponent, rotates similarly but in a CW direction.

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Examples

- **http://130.191.21.201/multimedia/jiracek/dga/spectralanalysis/examples.html**
- *The example clearly shows that the cosine function can be viewed as being composed of both positive (CCW) and negative (CW) frequency components.*

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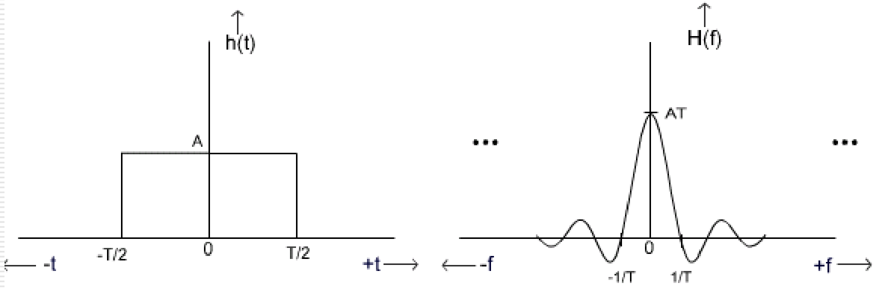
Famous FT pairs(1)

TIME DOMAIN | FREQUENCY DOMAIN



$h(t) = A \quad |t| < T/2$

$= A/2 \quad |t| = T/2$

$= 0 \quad |t| > T/2$

⟷

$H(f) = AT \dfrac{\sin(\pi T f)}{\pi T f}$

Boxcar function ↔Sinc Function

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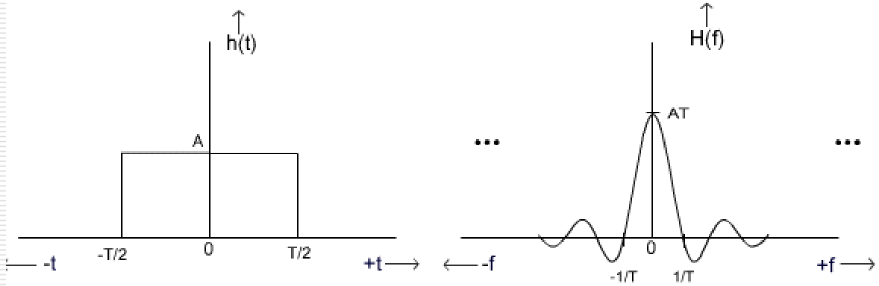
Famous FT pairs(2)

TIME DOMAIN | FREQUENCY DOMAIN



$h(t) = A \quad |t| < T/2$

$= A/2 \quad |t| = T/2$

$= 0 \quad |t| > T/2$

⟷

$H(f) = AT \dfrac{\sin(\pi T f)}{\pi T f}$

Constant Function ↔ Dirac Delta Function

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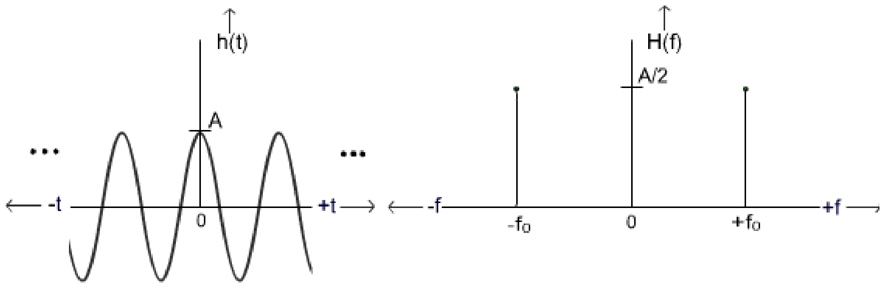
Famous FT pairs(3)

TIME DOMAIN | FREQUENCY DOMAIN



$h(t) = A\cos(2\pi f_0 t)$ ⟷ $H(f) = A/2 \ \delta(f - f_0) + A/2 \ \delta(f + f_0)$

Cosine Function ↔ Two real even delta function

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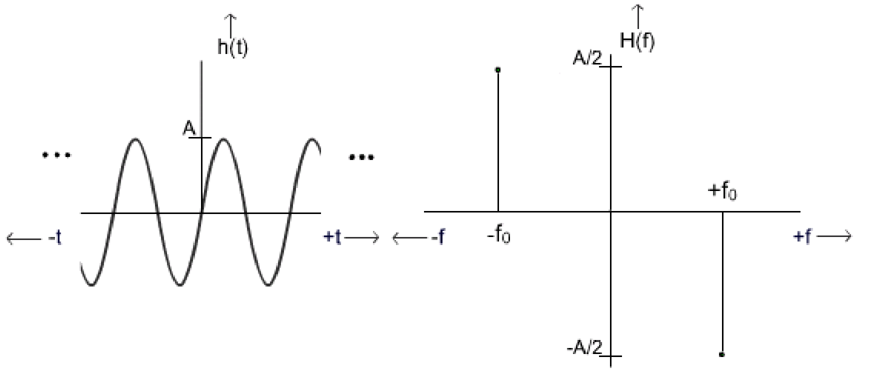
Famous FT pairs(3)

TIME DOMAIN | FREQUENCY DOMAIN



$h(t) = A\sin(2\pi f_0 t)$ ⟷ $H(f) = -i\,A/2 \ \delta(f - f_0) + i\,A/2 \ \delta(f + f_0)$

Sine Function ↔Two imagery odd delta functions

Two variable FT

- The FT can be easily extended to a function *f(x,y)* of two variables. If *f(x,y)* is continuous and integrable and *F(u,v)* is integrable, the following FT pair exist.

$$\Psi\{f(x,y)\} = F(u,v) = \int\int_{-\infty}^{+\infty} f(x,y) exp[-j2\pi(ux+vy)]dxdy$$

$$and$$

$$\Psi^{-1}\{F(u,v)\} = f(x,y) = \int\int_{-\infty}^{+\infty} F(u,v) exp[j2\pi(ux+vy)]dudv.$$

Here *u* and *v* are the frequency variables.

Two variable FT

- As in one dimensional case, the Fourier spectrum, phase and power spectrum respectively are:

$$F(u,v) = [R^2(u,v) + I^2(u,v)]^{1/2}$$

$$\phi(u,v) = tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$$

$$P(u,v) = |F(u,v)|^2 = R^2(u,v) + I^2(u,v)$$

Conclusion

- The Fourier transform is, in general, a complex function of the real frequency variables. As such the transform can be written in terms of its magnitude and phase.
- A 2D signal (an image) can also be complex and thus written in terms of its magnitude and phase.
- Both the magnitude and the phase functions are necessary for the complete reconstruction of an image from its Fourier transform.

Conclusion

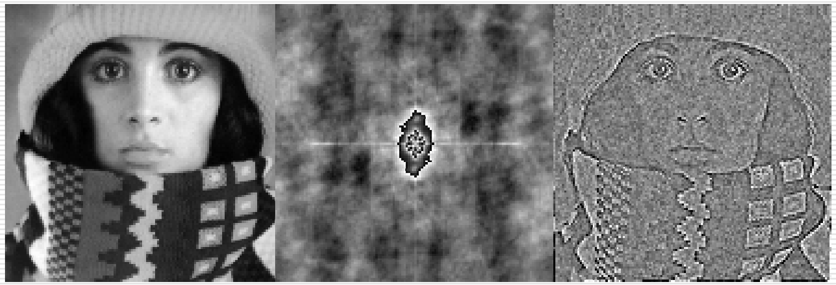


Figure 1b shows what happens when Figure 1a is restored solely on the basis of the magnitude information and Figure 1c shows what happens when Figure 1a is restored solely on the basis of the phase information

Discrete FT

1	2	3	4	5
2	1	8	0	5
3	5	7	5	9
0	2	2	3	0

Discrete FT

Discrete FT

- The DFT is the sampled Fourier Transform and therefore does not contain all frequencies forming an image, but only a set of samples which is large enough to fully describe the spatial domain image. The number of frequencies corresponds to the number of pixels in the spatial domain image, *i.e.* the image in the spatial and Fourier domain are of the same size.

Discrete FT

- The DFT is only usually defined for a discrete function *f(x,y)* that is nonzero only over a finite region,

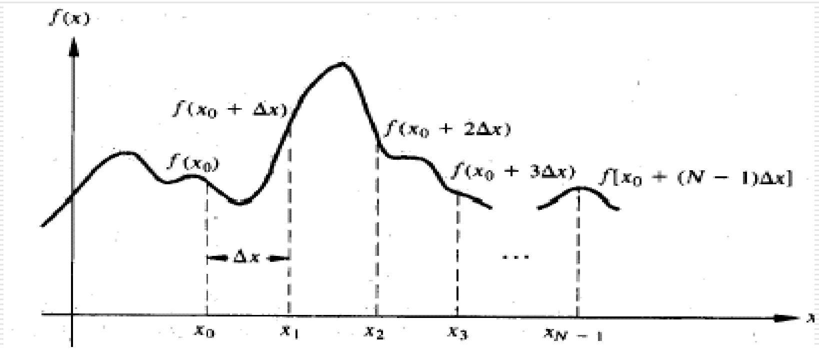
  0<=x<=M-1 and *0<=y<=N-1*,

  where *M* by *N* is the resolution of the spatial image.

Discrete FT (*1D Case*)

- Suppose that a continues function *f(x)* is discretized into a sequence by taking *N* samples *Δx* units apart

$$f(x_0), f(x_0+\Delta x), f(x_0+2\Delta x), ..., f(x_0+[N-1]\Delta x)$$



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Discrete FT (*1D Case*)

- It will be convenient in subsequent developments to use x as either a discrete or continues variable, depending on the context. To do so requires defining

$$f(x) = f(x_0 + x\Delta x)$$

where x now assumes the discrete values 0,1,2,3,….N-1.
In other words, the sequence {f(0), f(1), f(2),…., f(N-1)} denote any N uniformly spaced samples from a corresponding continue function.

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DFT Pairs

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)exp[-j2\pi ux/N],$$

$$for \quad u = 0, 1, 2, ...N-1$$

$$f(x) = \sum_{x=0}^{N-1} F(u)exp[j2\pi ux/N]$$

$$for \quad x = 0, 1, 2, ...N-1$$

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DFT Pairs

- The variable u =0,1,2,….N-1 in the DFT corresponds to the continues transform at values 0, Δu,…, (N-1) Δu. In other words, F(u) represents F(uΔu). This notation is similar to that used for the discrete f(x), except that the samples of F(u) start at the origin of the frequency axis. The term Δu and Δx are related by the expression

$$\Delta u = 1/(N\Delta x)$$

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DFT(2D Case)

$$F(u,v) = \frac{1}{NM}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)exp[-j\pi(ux/M + vy/N)],$$

$$for \quad u = 0, 1, 2, ...M-1, v = 0, 1, 2, ..., N-1$$

$$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} f(u,v)exp[j2\pi(ux/M + vy/N)],$$

$$for \quad x = 0, 1, 2, ...M-1, y = 0, 1, 2, ..., N-1$$

Δu=1/(MΔx) and Δv=1/(NΔy)

Conclusion

- A 2Dfunction is represented in a computer as numerical values in a matrix, whereas a one-dimensional Fourier transform in a computer is an operation on a vector.
- A 2-D Fourier transform can be computed by a sequence of 1-D Fourier transforms. We can first transform each column vector of the matrix and then each row vector of the matrix. Alternately, we can first do the rows and later do the columns.

Separability

- The discrete FT pair can be expressed in separable forms which (after some manipulations) can be expressed as:

$$F(u,v) = \frac{1}{M}\sum_{x=0}^{M-1} F(x,v)\exp[-j2\pi ux/M]$$

Where: $$F(x,v) = \left[\frac{1}{N}\sum_{y=0}^{N-1} f(x,y)\exp[-j2\pi vy/N]\right]$$

Translation (shifting)

$$f(x,y)\exp[j2\pi(u_0 x/M + v_0 y/N)] \Leftrightarrow F(u-u_0, v-v_0)$$

and

$$f(x-x_0, y-y_0) \Leftrightarrow F(u,v)\exp[-j2\pi(ux_0/M + vy_0/N)]$$

Periodicity & Conjugate Symmetry

- The discrete FT and its inverse are periodic with period N:

$$F(u,v)=F(u+M,v)=F(u,v+N)=F(u+M,v+N)$$

=> Only one period of the transform is necessary to specify F(u,v) completely in the frequency domain.

Periodicity & Conjugate Symmetry

- For real f(x,y), FT also exhibits conjugate symmetry:

$$F(u,v) = F^*(-u,-v)$$

or $$|F(u,v)| = |F(-u,-v)|$$

=> The magnitude of the transform is centered on the origin.

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Rotation

- Polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad u = \omega\cos\varphi, \quad v = \omega\cos\varphi$$

Which means that:

$$f(x,y), F(u,v) \text{ become } f(r,\theta), F(\omega,\varphi)$$

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Rotation

$$f(r,\theta+\theta_0) \Leftrightarrow F(\omega,\varphi+\theta_0)$$

=> Which means that rotating f(x,y) by an angle θ_0 rotates F(u,v) by the same angle (and vice versa).

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Distributivity and Scaling

- Distributivity only stands for addition

$$\mathcal{F}\{f_1(x,y)+f_2(x,y)\} = \mathcal{F}\{f_1(x,y)\}+\mathcal{F}\{f_2(x,y)\}$$

$$\mathcal{F}\{f_1(x,y)\cdot f_2(x,y)\} \neq \mathcal{F}\{f_1(x,y)\}\cdot\mathcal{F}\{f_2(x,y)\}$$

- For two scalars *a* and *b*

$$af(x,y) \Leftrightarrow aF(u,v)$$

$$f(ax,by) \Leftrightarrow \frac{1}{|ab|}F(u/a,v/b)$$

Average value

- An average value of a 2-D discrete function f(x,y) is related to the Fourier transform:

$$\overline{f}(x,y) = \frac{1}{N} F(0,0)$$

Laplacian

- The Laplacian operator is useful for finding edges in an image.

$$\mathcal{F}\{\nabla^2 f(x,y)\} \Leftrightarrow -(2\pi)^2(u^2 + v^2)F(u,v)$$

Convolution and correlation

- Convolution

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt$$

- In image analysis f or g is image and the other one is the neighborhood weighting function (Morse).

Convolution and correlation

- Convolution theorem

$$f(x,y) * g(x,y) \Leftrightarrow F(u,v)G(u,v)$$

$$f(x,y)g(x,y) \Leftrightarrow F(u,v) * G(u,v)$$

- Correlation theorem

$$f(x,y) * g(x,y) \Leftrightarrow F(u,v)G(u,v)$$

$$f(x,y)g(x,y) \Leftrightarrow F(u,v) * G(u,v)$$

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Sampling

- To establish the sampling conditions so that a continuous image can be restored fully.
- Band limited signal: F(s)=0 for |s|>2πW
- Sampling interval

$$\Delta x \le \frac{1}{2W}$$

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Fast Fourier Transform

FFT is simply a fast (computationally efficient) way to calculate the Discrete Fourier Transform (DFT).

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Fast Fourier Transform

Reduces computational complexity of DFT algorithm from class $O(n^2)$ to class O(n log n).

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Fast Fourier Transform

First FFT algorithm was developed by J.W. Cooley and J.W. Tukey. Published in April, 1965. It uses divide-and-conquer paradigm and is based on the symmetry properties of the complex exponential in the definition of the DFT.

Fast Fourier Transform

Implemented with many modifications, depending on the computational needs.

Mainly used algorithms:

- Radix-2 (Cooley-Tukey)
- Radix-4
- Split radix
- FHT (Fast Hartley Transform)
- QFT
- DITF

Fast Fourier Transform

There is no need to implement the FFT yourself. Many good FFT implementations are available in different languages. Microprocessor manufacturers generally provide optimized FFT implementation in their processors' assembly code.

Fast Fourier Transform

References

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